Question 1

1. Simplify the following expression for x

$$x = \log_3 81 + \log_3 \frac{1}{9} \ .$$

- $x = \log_3 81 + \log_3 \frac{1}{9} = \log_3 \frac{81}{9}$
- $= \log_3 9 = \log_3 3^2 = 2 \log_3 3 = 2.$

Question 2

2. The function $f(x) = x^3 + 3x + e^{2x}$ is one-to-one. Compute $(f^{-1})'(1)$.

- $(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$
- By trial-and-error we determine that $f^{-1}(1) = 0$ (that is $f(0) = 1$). $f'(x) = 3x^2 + 3 + 2e^{2x}$.
- Hence $f'(f^{-1}(1)) = f'(0) = 5$.
- Therefore $(f^{-1})'(1) = \frac{1}{5}$.

Question 3

3. Differentiate the function

$$f(x) = \frac{(x^2-1)^4}{\sqrt{x^2+1}}.$$

- Use logarithmic differentiation. (Take logarithm of both sides of equation, then do implicit differentiation.)
- $y = \frac{(x^2-1)^4}{\sqrt{x^2+1}}$.
- $\ln y = 4\ln(x^2-1) - \frac{1}{2}\ln(x^2+1)$
- $\frac{1}{y}\frac{dy}{dx} = \frac{8x}{x^2-1} - \frac{x}{x^2+1}$.
- $f'(x) = \frac{dy}{dx} = \frac{x(x^2-1)^4}{\sqrt{x^2+1}}\left(\frac{8}{x^2-1} - \frac{1}{x^2+1}\right)$.

Question 4

4. Compute the integral

$$\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} dx.$$

- Make the substitution $u = \ln \frac{x}{2}$ with $dx = xdu$. At $x = 2e$, have $u = 1$ and at $x = 2e^2$ have $u = 2$.
- $\int_{2e}^{2e^2} \frac{1}{x(\ln \frac{x}{2})^2} dx = \int_1^2 \frac{1}{u^2} du = \left[-\frac{1}{u}\right]_1^2$
- $= -\frac{1}{2} + 1 = \frac{1}{2}$.

Question 5

5. Which of the following expressions gives the partial fraction decomposition of the function

$$f(x) = \frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)}?$$

- We have x is a linear factor of multiplicity 3, $(x-3)$ is a linear factor of multiplicity 1 and (x^2+4) is an irreducible quadratic factor of multiplicity 1.
- $$\frac{x^2 - 2x + 6}{x^3(x-3)(x^2+4)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-3} + \frac{Ex+F}{x^2+4}.$$

Question 6

6. Find $f'(x)$ if
$$f(x) = x^{\ln x} .$$

- One method is to use logarithmic differentiation. Let $y = f(x)$.
- $\ln y = \ln(x^{\ln x}) = (\ln x)(\ln x) = (\ln x)^2$.
- $\frac{y'}{y} = \frac{2\ln x}{x}$.
- Therefore $f'(x) = y' = 2(\ln x)x^{(\ln x)-1}$.
- **Alternatively** we have $f(x) = \left(e^{\ln x}\right)^{\ln x} = e^{(\ln x)^2}$
- $f'(x) = e^{(\ln x)^2}\frac{d(\ln x)^2}{dx} = e^{(\ln x)^2}(2\ln x)\frac{1}{x} = \frac{x^{\ln x}2\ln x}{x} = x^{(\ln x)-1}2\ln x$.

Question 7

7. Calculate the following integral

$$\int_0^1 \frac{\arctan x}{1+x^2}\ dx\ .$$

- Make the substitution $u = \arctan x$ with $dx = (1+x^2)du$.
- $\int_0^1 \frac{\arctan x}{1+x^2}\ dx = \int_0^{\frac{\pi}{4}} u\ du$
- $= \left[\frac{u^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}$.

Question 8

8. Evaluate the integral

$$\int_0^{\pi/2} \sin^3(x)\cos^5(x)dx.$$

- Use the identity $1 - \cos^2(x) = \sin^2(x)$.
- Let $u = \cos(x),\ du = -\sin(x)dx$
- $\int_0^{\pi/2} \sin^3(x)\cos^5(x)dx = \int_0^{\pi/2}(1-\cos^2(x))\sin(x)\cos^5(x)dx = -\int_1^0 (u^5 - u^7)\ du.$
- $= \int_0^1 (u^5 - u^7)\ du = \left[\frac{u^6}{6} - \frac{u^8}{8}\right]_0^1$
- $= \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$

Question 9

9. *Compute the limit*

$$\lim_{x\to 2}\left(\frac{x}{2}\right)^{\frac{1}{x-2}}.$$

- We have an indeterminate form 1^{∞}.
- Let $L = \lim_{x\to 2}\left(\frac{x}{2}\right)^{\frac{1}{x-2}}$. Then

$$\ln L = \lim_{x\to 2}\ln\left(\frac{x}{2}\right)^{\frac{1}{x-2}} = \lim_{x\to 2}\frac{\ln\left(\frac{x}{2}\right)}{x-2} \quad \text{(l'Hospital's rule)} \ = \lim_{x\to 2}\frac{1}{x} = \frac{1}{2}.$$

- Therefore $L = e^{\frac{1}{2}} = \sqrt{e}$.

Question 10

10. Evaluate the integral

$$\int x^2 \cos(2x)dx.$$

- We use integration by parts with $u = x^2$ and $dv = \cos(2x)dx$, so $du = 2xdx$ and $v = \frac{1}{2}\sin(2x)$.
- $\int udv = uv - \int vdu$
  $\int x^2 \cos(2x)dx = \frac{1}{2}x^2 \sin(2x) - \int x \sin(2x)dx$
- (we use integration by parts again with $u_1 = x, \quad dv_1 = \sin(2x)dx$, $du_1 = dx, \quad v_1 = \frac{-\cos(2x))}{2}$.)
- $= \frac{1}{2}x^2 \sin(2x) - \left[-\frac{1}{2}x\cos(2x) - \int(-\frac{1}{2}\cos(2x))dx\right]$
- $= \frac{1}{2}x^2 \sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C$

Question 11

11. *Evaluate*

$$\int \frac{1}{3}x^3\sqrt{9-x^2}\;dx.$$

- two approaches work: trigonometric substitution with $x = 3\sin\theta$ or alternatively u substitution with $u = 9 - x^2$. The method of trigonometric substitution is outlined here, although the latter method may be somewhat easier.
- $(x = 3\sin\theta,\ dx = 3\cos\theta d\theta)$ $\int \frac{1}{3}x^3\sqrt{9-x^2}\;dx = \int 81\sin^3\theta\cos^2\theta d\theta.$
- $= \int 81(1-\cos^2\theta)\sin\theta\cos^2\theta d\theta \quad (u = \cos\theta,\ du = -\sin\theta d\theta)$
- $= \int 81(u^4 - u^2)du$
- $= \frac{81\cos^5\theta}{5} - 27\cos^3\theta + C \quad \left(\cos\theta = \frac{1}{3}\sqrt{9-x^2}\ \ \text{[from triangle]}\right)$
- $= \frac{(9-x^2)^{\frac{5}{2}}}{15} - (9-x^2)^{\frac{3}{2}} + C.$

Question 12

12. Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = kC(t)$, where k is a constant. The initial concentration of the drug is 4 mg/ml. After 5 hours, the concentration is 3 mg/ml.

- (a) Give a formula for the concentration of the drug at time t.
- $C(t) = C(0)e^{kt} = 4e^{kt}$
- $C(5) = 3 = 4e^{k5}$
- We solve for k, $k = \frac{1}{5}\ln\left(\frac{3}{4}\right)$
- Substituting into $C(t)$, we get $C(t) = 4\left(\frac{3}{4}\right)^{\frac{1}{5}t}$
- (b) How much drug will there be in 10 hours?
- $C(10) = 4\left(\frac{3}{4}\right)^2 = \frac{9}{4}$
- (c) How long will it take for the concentration to drop to 0.5 mg/ml?
- $C(t) = 4\left(\frac{3}{4}\right)^{\frac{1}{5}t} = \frac{1}{2}$
- We solve for t, $\left(\frac{3}{4}\right)^{\frac{1}{5}t} = \frac{1}{8} \quad \rightarrow \quad \frac{t}{5}\ln\left(\frac{3}{4}\right) = \ln\left(\frac{1}{8}\right) = -\ln 8$
  $\rightarrow \quad t = \frac{-5\ln 8}{\ln 3 - \ln 4}$.